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THE STABILITY AND STABILIZATION OF THE STEADY MOTIONS OF A CLASS OF NON-HOLONOMIC MECHANICAL SYSTEMS†

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The stability of the steady motions and the controllability of a class of non-holonomic mechanical systems under the action of potential and control forces are investigated. A problem of the stability of the steady motion of a three-wheeled vehicle, taking into account the inertia of the wheels, which is an example of systems of this class, is considered.

1. STEADY MOTIONS

Consider a non-holonomic mechanical system, the position of which is defined by the generalized coordinates $q_1, \dots, q_n$. We shall assume that is possesses certain characteristics. In fact, its generalized coordinates can be chosen so that the following conditions are satisfied.

The velocities $\dot{q}_1, \dots, \dot{q}_n$ are restricted by $n - l(l < n)$ steady non-holonomic constraints which can be represented in the form of the two groups

$$\dot{q}_\chi = \sum_{r=1}^{l} b_{\chi r}(q)\dot{q}_r \tag{1.1}$$

$$\dot{q}_\rho = \sum_{r=1}^{l} b_{\rho r}(q)\dot{q}_r \tag{1.2}$$

Hereforth, the subscripts take the values

$$p, r, s = 1, \dots, l;\ \chi = l+1, \dots, m;\ \rho = m+1, \dots, n;\ \mu = l+1, \dots, n.$$

Elimination of the quantities $\dot{q}_\chi$, $\dot{q}_\rho$ using the equations of the constraints (1.1) and (1.2) from the expressions for T and $\partial T/\partial \dot{q}_\mu$ (T is the kinetic energy of the system) leads to the expressions

$$2\Theta = \sum_{r,s=1}^{l} a_{rs}(q)\dot{q}_r\dot{q}_s > 0, \quad \Theta_\mu = \sum_{p=1}^{l} \Theta_{\mu p}(q)\dot{q}_p \tag{1.3}$$

Also, suppose the following conditions are satisfied:
(1) the coefficients $b_{\chi r}$ in Eqs (1.1) are functions solely of the coordinates $q_{l+1}, \dots, q_m$, the velocities of which are dependent by virtue of the same Eqs (1.1), while the coefficients $b_{\rho r}$ in Eqs (1.2) may depend on the coordinates $q_1, \dots, q_l, q_{l+1}, \dots, q_m$;
(2) the potential forces acting on the system are the derivatives of a force function $U(q)$ which also depends on the coordinates q_χ, that is, $U = U(q_\chi)$;

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(3) the coefficients a_{rs} in expression (1.3) and the expression

$$\sum_{\mu=l+1}^{n} \Theta_{\mu p} \nu_{\mu rs}$$

where

$$\nu_{\mu rs} = \frac{\partial b_{\mu r}}{\partial q_s} - \frac{\partial b_{\mu s}}{\partial q_r} - \sum_{\mu'=l+1}^{n} \left(b_{\mu' r} \frac{\partial b_{\mu s}}{\partial q_{\mu'}} - b_{\mu' s} \frac{\partial b_{\mu r}}{\partial q_{\mu'}} \right)$$

depend solely on the coordinates $q_{l+1}, \ldots, q_m$.

In the case of a non-holonomic mechanical system belonging to the class under consideration, the equations of motion in the form of Voronets equations [1, 2] have the form

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_r} = \sum_{\chi=l+1}^{m} \frac{\partial(\Theta + U)}{\partial q_\chi} b_{\chi r} + \sum_{\mu=l+1}^{n} \sum_{p,s=1}^{l} \Theta_{\mu p} \nu_{\mu rs} \dot{q}_s \dot{q}_p + Q_r + \sum_{\chi=l+1}^{m} Q_\chi b_{\chi r} \tag{1.4}$$

where Q_r, Q_χ are control forces, which we shall assume to depend solely on the variables $\dot{q}_r$, q_χ.

A specific characteristic of systems of this class is the fact that, when there are no controls ($Q_r = 0, Q_\chi = 0$), all the coordinates q_r are cyclic in the sense of the definition in [2–4] and the coordinates q_χ ($\chi = l+1, \ldots, m$) are positional.

It should be emphasized that, unlike in the general case, Eqs (1.4), together with the equations of the constraints (1.1), form a closed system of *first* order equations in $\dot{q}_r$, q_χ and do not contain the coordinates q_r explicitly. The equations for the non-holonomic constraints (1.2) are equation for constraints of the Chaplygin type.

We shall assume that, for certain initial conditions and $Q_r = 0$, $Q_\chi = 0$, steady motion of the system is possible, during which the positional coordinates and the cyclic velocities are constant:

$$\dot{q}_r(t) = \dot{q}_{r0} = \omega_r \quad q_\chi(t) = q_{\chi 0} \tag{1.5}$$

In this case, the m constant quantities ω_r, $q_{\chi 0}$ satisfy the m equations

$$\sum_{p,s=1}^{l} \left\{ \sum_{\mu=l+1}^{n} \Theta_{\mu p} \nu_{\mu rs} + \sum_{\chi=l+1}^{m} \frac{1}{2} b_{\chi r} \frac{\partial a_{ps}}{\partial q_\chi} \right\}_0 \omega_p \omega_s + \sum_{\chi=l+1}^{m} \left\{ b_{\chi r} \frac{\partial U}{\partial q_\chi} \right\}_0 = 0 \tag{1.6}$$

$$\sum_{r=1}^{l} (b_{\chi r})_0 \omega_r = 0 \tag{1.7}$$

A zero subscript denotes that an expression is calculated for the values of the variables corresponding to the steady motion (1.5).

Depending on the parameters of system (1.6), (1.7), there can be one or several isolated solutions. Cases are possible when only $m - m'$ independent equations can be found among Eqs (1.6) and (1.7) and the system being considered will then have a set of steady motions of the form (1.5) of dimensionality m'.

Note that, when the conditions for the existence of steady motions, which were formulated earlier for the general case in [2–4], are satisfied, Eqs (1.6) and (1.7) are satisfied identically with respect to ω_r, $q_{\chi 0}$, that is, an m-dimensional manifold of steady motions exists.

2. INVESTIGATION OF THE STABILITY

We choose an arbitrary point ω_r, $q_{\chi 0}$, defined by relations (1.6) and (1.7), and consider the question of the stability of solution (1.5) of the system of equations (1.1), (1.4) with respect to perturbations of the variables $\dot{q}_r$, q_χ.

We introduce the deviations

$$y_r = \dot{q}_r - \omega_r, \quad z_\chi = q_\chi - q_{\chi_0}$$

The equations of the perturbed motion in the variables

$$\underset{(l\times 1)}{y} = \|y_1 \dots y_l\|^T, \quad \underset{((m-l)\times 1)}{z} = \|z_{l+1} \dots z_m\|^T$$

have the form

$$A\dot{y} = P_1 y + V_1 z + F^{(1)}u^{(1)} + P_2^T F^{(2)}u^{(2)} + Y(y, z), \quad \dot{z} = P_2 y + V_2 z + Z(y, z) \tag{2.1}$$

The elements of the matrices A, P and V are calculated in a similar manner to that indicated earlier in [2]; $Y(y, z, u^{(1)}, u^{(2)}, Z(y, z)$ are vector functions which contain terms of higher than the first order in the variables y, z which have been introduced and $\underset{(l\times 1)}{F^{(1)}u^{(1)}}$, $\underset{((m-l)\times 1)}{F^{(2)}u^{(2)}}$ are the linear parts of the control functions.

Unlike the equations of the perturbed motion of a non-holonomic system of general form, which has been considered earlier in [4–7], the system of equations (2.1) is a first-order system and does not contain a block of equations corresponding to the positional coordinates. For this reason, the problem of the stability of the steady motion of a non-holonomic mechanical system of this class cannot be reduced to a problem on the stability of the equilibrium position of a certain holonomic system, and the theorems proved in [2–7] cannot be applied to it. This fact also actually justifies the advisability of separating out the special of non-holonomic systems introduced above.

The characteristic equation of the linearized homogeneous systems (2.1) has the form

$$\Delta(\lambda) = \det G(\lambda) = 0, \quad G(\lambda) = \begin{Vmatrix} A\lambda - P_1 & -V_1 \\ -P_2 & \lambda E - V_2 \end{Vmatrix} \tag{2.2}$$

If a non-holonomic mechanical system possesses a manifold of steady motions of dimensionality m', then the condition $\operatorname{rank} G(0) \le m - m'$ is satisfied. If $\operatorname{rank} G(0) = m - m'$, then the characteristic equation (2.2) has m' zero roots, which corresponds to the Lyapunov critical case, and the Lyapunov–Malkin theorem [8, 9] can be used to investigate the stability of the steady motion.

The following assertion can therefore be formulated in the case of systems belonging to the class being considered when there are no controls.

Theorem 1. If the non-holonomic system (1.1), (1.4), when conditions 1–3 of section 1 are satisfied, has a manifold of steady motions defined by relation (1.6) and (1.7), then the steady motion (1.5) is stable (unstable) if all of the roots of Eqs (2.2), apart from the m' zero roots, have negative real parts (at least one root with a positive real part). In the case of stability, every perturbed motion, which is sufficiently close to the unperturbed motion, tends, when $t \to \infty$, to one of the possible steady motions belonging to the above-mentioned manifold, defined by relations (1.6) and (1.7).

If the additional conditions [2–4] are satisfied, the matrices P_1, P_2, V_1, V_2 are null matrices and the number of zero roots of the characteristic equation (2.2) is equal to m. In this case, it is necessary to consider the complete non-linear system when analysing the stability.

3. CONTROLLABILITY

The criteria for controllability and observability were formulated earlier in [5] for non-holonomic mechanical systems of general form. Using the criterion in [10], it is easy to obtain the criteria of controllability for non-holonomic systems of the class being considered.

Theorem 2. System (2.1) is controllable when and only when

$$\operatorname{rank} G_1 = m, \quad \forall \lambda \in \Lambda, \quad \Lambda = \{\lambda_i : \det G(\lambda) = 0\} \tag{3.1}$$

when

$$G_1 = \| G(\lambda) \; F \|, \quad F = \begin{Vmatrix} F^{(1)} & P_2^T F^{(2)} \\ 0 & 0 \end{Vmatrix}$$

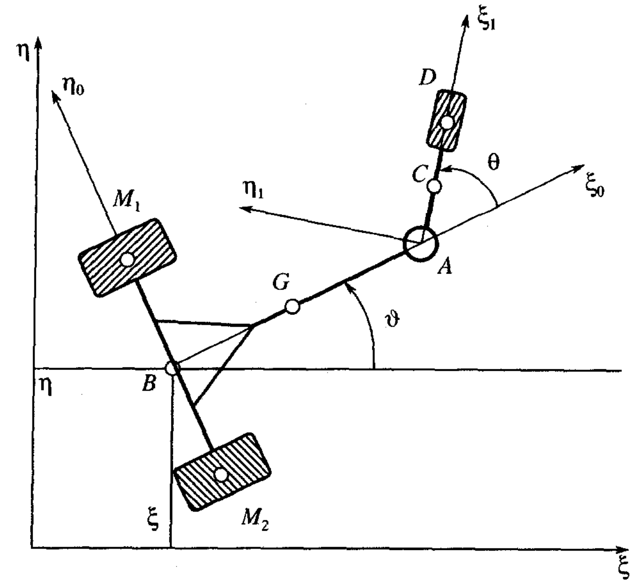

Fig. 1

Corollary. 1. If the control act solely on all of the cyclic coordinates ($F^{(1)} = E_1$, ($F^{(2)} = 0$)), then system (2.1) is controllable when and only when rank$\|P_2 \lambda E - V_2\| = m - l$, $\forall\lambda \in \Lambda$.
2. If the controls are only introduced via all the positional coordinates ($F^{(1)} = 0, F^{(2)} = E_{m-1}$) and the system has a manifold of steady motions of dimensionality m', then, for system (2.1) to be controllable, it is necessary that the dimensionality of the manifold of steady motions should not exceed the number of positional coordinates ($m - l$).

The last assertion shows that, if $m' > m - l$, then, in order to stabilize the steady motion, it is necessary to introduce controls not only through the positional coordinates but just through some of the cyclic coordinates.

The criteria of observability for the systems being considered, when there is information of one form or another, can be obtained in a similar manner.

4. THE STEADY MOTIONS OF A THREE-WHEELED VEHICLE

We will consider a model of a three-wheeled vehicle (a tricycle) as a system of rigid bodies: a trolley of mass m_T, the body of this trolley is rigidly coupled to an axle onto which two wheels of radius r are fitted with masses m_1 and m_2 (with centres at the points M_1 and M_2), a vertical strut of mass m_c (with centre at the point M_c), coupled to the trolley by a vertical hinge at the point A, a rigid axle is fastened at point D of the strut and this axle is fitted with a wheel of radius R and mass m_3 (with centre at the point M_3).

The wheels roll over a rough horizontal plane without sliding and without leaving the surface. We neglect the displacement of the centre of mass of the system which arises when the leading part of the tricycle is rotated [11].

We introduce a fixed system of coordinates $O\xi\eta\xi$. The body and the strut execute plane-parallel motion in the horizontal plane $O\xi\eta$. We denote the projections of the centres of mass of the strut, the trolley and the third wheel onto the plane $O\xi\eta$ by C, G and D and define the position of the system by means of the coordinates ξ, η, θ, ϑ, φ_1, φ_2, φ_3 (see Fig. 1): ξ and η are the coordinates of point B, the middle of the rear bridge in the system of coordinates $O\xi\eta$, υ is the angle between the axis of symmetry of the trolley AB and the fixed axis $O\xi$, and θ is the angle of rotation of the strut about the axis of the trolley AB; here $\vartheta_1 = \theta + \vartheta$, where ϑ_1 is the angle of rotation of the strut about the fixed axis $O\xi$, and φ_1, φ_2, φ_3 are the angles of rotation of the wheels around the corresponding axes.

We introduce the following notation

$$a = M_1B = M_2B, \quad l_1 = BG, \quad l_2 = GA, \quad l = l_1 + l_2 = AB, \quad d = AC, \quad b = AD$$

The conditions that there is no slip of the wheels in this problem means that there are no components of the velocities of the points of contact of the wheels with the plane of rolling in the transverse and longitudinal directions.

In the notation adopted, these conditions (the conditions of non-holonomic constraints) take the form

$$\begin{aligned}
&\dot{\xi}\cos\vartheta + \dot{\eta}\sin\vartheta - a\dot{\vartheta} - r\dot{\varphi}_1 = 0, \quad \dot{\xi}\sin\vartheta - \dot{\eta}\cos\vartheta = 0 \\
&\dot{\xi}\cos\vartheta + \dot{\eta}\sin\vartheta + a\dot{\vartheta} - r\dot{\varphi}_2 = 0 \\
&\dot{\xi}\cos\vartheta_1 + \dot{\eta}\sin\vartheta_1 + l\dot{\vartheta}\sin\theta - R\dot{\varphi}_3 = 0 \\
&-\dot{\xi}\sin\vartheta_1 + \dot{\eta}\cos\vartheta_1 + l\dot{\vartheta}\cos\theta + b\dot{\vartheta}_1 = 0
\end{aligned} \tag{4.1}$$

We introduce new generalized coordinates, which also uniquely define the position of the mechanical system

$$q_1 = \varphi_1 + \varphi_2, \quad q_2 = \vartheta, \quad q_3 = \theta, \quad q_4 = \xi, \quad q_5 = \eta, \quad q_6 = \varphi_3, \quad q_7 = \varphi_1 - \varphi_2 \tag{4.2}$$

The equations of non-holonomic constraints (4.1) take a simpler form in the variables (4.2), where one constraints is a non-holonomic constraint of the general form of (1.1)

$$\dot{q}_3 = b_{31}\dot{q}_1 + b_{32}\dot{q}_2 \tag{4.3}$$

and the remaining constraints of (1.2) are constraints of the Chaplygin type

$$\dot{q}_4 = b_{41}\dot{q}_1, \quad \dot{q}_5 = b_{51}\dot{q}_1, \quad \dot{q}_6 = b_{61}\dot{q}_1 + b_{62}\dot{q}_2, \quad \dot{q}_7 = b_{71}\dot{q}_2 \tag{4.4}$$

Here

$$\begin{aligned}
&b_{31}(q_3) = \frac{r}{b}\sin q_3, \quad b_{32}(q_3) = -\left(1 + \frac{l}{b}\cos q_3\right), \quad b_{41}(q_2) = r\cos q_2 \\
&b_{51}(q_2) = r\sin q_2, \quad b_{61}(q_3) = \frac{r}{R}\cos q_3, \quad b_{62}(q_3) = \frac{l}{R}\sin q_3, \quad b_{72} = \frac{2a}{r}
\end{aligned} \tag{4.5}$$

The coefficients b_{31} and b_{32}, being solely functions of the coordinate q_3, satisfy condition 1 of Section 1.

The expression for the kinetic energy of the system ignoring constraints (4.3) and (4.4) has the form

$$T = \frac{1}{2}\sum_{i,j}^{7} A_{ij}\dot{q}_i\dot{q}_j$$

Here

$$A_{11} = A_{77} = J, \quad A_{17} = \frac{1}{2}J_4, \quad A_{22} = I + I_4 + 2S_3 l\cos q_3, \quad A_{23} = \frac{1}{2}(I_4 + S_3 l\cos q_3)$$

$$A_{24} = \frac{1}{2}(-S_1\sin q_2 + S_2\cos q_2 - S_3\sin(q_2 + q_3))$$

$$A_{25} = \frac{1}{2}(S_1\cos q_2 + S_2\sin q_2 + S_3\cos(q_2 + q_3))$$

$$A_{33} = I_4, \quad A_{34} = -\frac{1}{2}S_3\sin(q_2 + q_3), \quad A_{35} = \frac{1}{2}S_3\cos(q_2 + q_3)$$

$$A_{44} = A_{55} = M, \quad A_{66} = J_3$$

The remaining coefficients A_{ij} are equal to zero. Here

$$J = \frac{1}{4}(J_1 + J_2), \quad J_4 = \frac{1}{4}(J_1 - J_2), \quad M = m_T + m_C + m_1 + m_2 + m_3$$

$$S_1 = m_T l_1 + (m_C + m_3)l, \quad S_2 = (m_2 - m_1)a, \quad S_3 = m_C d + m_3 b$$

$$I = m_T l_1^2 + (m_1 + m_2)a^2 + (m_C + m_3)l^2 + I_1 + I_2 + J_G$$

$$I_4 = m_C d^2 + m_3 b^2 + J_C + I_3$$

$J_j (j = 1, 2, 3)$ and $I_i (i = 1, 2)$ are the moments of inertia of the wheels about their axes of rotation and diameters, respectively, and J_G and J_C are the moments of inertia of the trolley and the strut about the vertical axes passing through their centres of mass.

Eliminating the quantities $\dot{q}_3, \dots, \dot{q}_7$ from the expression for the kinetic energy T using the equations of the constraints (4.3), (4.4), we obtain an expression for the reduced kinetic energy

$$\Theta = \frac{1}{2}[a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2]$$

Here

$$a_{11} = \overset{0}{a}_{11} + \overset{1}{a}_{11}\cos^2 q_3 + \overset{2}{a}_{11}\sin^2 q_3 = Mr^2 + J + \frac{r^2}{R^2}J_3\cos^2 q_3 + \frac{r^2}{b^2}I_5\sin^2 q_3$$

$$a_{12} = \overset{0}{a}_{12} + \overset{1}{a}_{12}\sin q_3\cos q_3 = S_2 r + \frac{2a}{r}J_4 + rl\mu\sin q_3\cos q_3$$

$$a_{22} = \overset{0}{a}_{22} + \overset{1}{a}_{22}\cos^2 q_3 + \overset{2}{a}_{22}\sin^2 q_3 = I + \frac{4a^2}{r^2}J + \frac{l^2}{b^2}I_5\cos^2 q_3 + \frac{l^2}{R^2}J_3\sin^2 q_3$$

$$I_5 = I_4 - 2bS_3, \quad \mu = \frac{J_3}{R^2} - \frac{1}{b^2}I_5$$

The coefficients a_{11}, a_{22} and a_{33} depend solely on the coordinate q_3 (the angle θ)

In this case, the three-subscript symbols $\nu_{\chi rs}$ ($\chi = 3, \dots, 7; r = 1, 2; s = 1, 2$) have the form

$$\nu_{311} = 0, \quad \nu_{312} = -\nu_{321} = -\frac{r}{b^2}(b^2\cos q_3 + 1)$$

$$\nu_{412} = -\nu_{421} = -r\sin q_2, \quad \nu_{712} = \nu_{721} = 0$$

$$\nu_{512} = -\nu_{521} = r\cos q_2, \quad \nu_{612} = -\nu_{621} = \frac{r}{R}\sin q_3$$

It can be shown that the expression $\sum_{\chi=3}^{7} \Theta_{\chi p}\nu_{\chi rs}$ also depends solely on the coordinate q_3, that is, it satisfies condition 3 of Section 1.

Hence, the mechanical model describing the motion of the tricycle is an example of a non-holonomic systems, which belongs to the class distinguished above.

The equations of motion of the tricycle in the form of the Voronets equations (1.5) have the form

$$\begin{aligned} \frac{d}{dt}(a_{11}\dot{q}_1 + a_{12}\dot{q}_2) &= \dot{q}_2(\beta_1\dot{q}_1 + \beta_2\dot{q}_2) + b_{31}\frac{\partial\Theta}{\partial q_3} + Q_1 + b_{31}Q_3 + \sum_{i=4}^{6} b_{i1}Q_i \\ \frac{d}{dt}(a_{12}\dot{q}_1 + a_{22}\dot{q}_2) &= -\dot{q}_1(\beta_1\dot{q}_1 + \beta_2\dot{q}_2) + b_{32}\frac{\partial\Theta}{\partial q_3} + Q_2 + b_{32}Q_3 + \sum_{i=6,7} b_{i2}Q_i \end{aligned} \tag{4.6}$$

Here

$$\beta_1 = \beta_1(q_3) = r^2 \sin q_3 \left[\mu \cos q_3 - \frac{1}{b^3}\gamma\right]$$

$$\beta_2 = \beta_2(q_3) = \left(\frac{lr}{R^2}J_3 + S_1 r\right) + lr\cos q_3 \left[\mu \cos q_3 - \frac{1}{b^3}\gamma\right]$$

$$\frac{\partial\theta}{\partial q_3} = \mu\left[\frac{1}{2}(l^2\dot{q}_1^2 - r^2\dot{q}_2^2)\sin 2q_3 + lr\dot{q}_1\dot{q}_2\cos 2q_3\right], \quad \gamma = I_4 - bS_3$$

where Q_i ($n = 1, \dots, 7$) are the control functions.

The coordinates $q_1 = \varphi_1 + \varphi_2$ (the sum of the angles of rotation of the wheels) and $q_2 = \vartheta$ (the angle of the track) are cyclic [2, 3], and the coordinate $q_3 = \theta$ is positional.

It should be emphasized that, in this problem, the introduction of generalized coordinates using formulae (4.2) enabled us to obtain the equations of motion of the non-holonomic system in a fairly simple form. The equations of motion (4.6) and the equation of the constraint (4.3) form a closed system of first-order equations in the variables $\dot{q}_1$, $\dot{q}_2$, q_3 and do not explicitly contain the coordinates q_1 and q_2; the remaining variables are found from the equations of the constraints (4.4). It is obvious that the analysis of the steady motions of the system, the analysis of their stability and, also, the controllability and observability of the system are simplified using this approach.

The system of equations (4.6), (4.3), with certain initial conditions and when there are no controls, admits of the particular solution

$$\dot{q}_1(t) = \dot{q}_{10} = \omega; \quad \dot{q}_2(t) = \dot{q}_{20} = \Omega; \quad q_3(t) = q_{30} = \theta_0 \tag{4.7}$$

which describes the steady motions of the system

The parameters defining the steady motions of the system, satisfy the conditions

$$\begin{aligned} &(\beta_1^0\omega + \beta_2^0\Omega)\Omega + rb^{-1}K_0\sin\theta_0 = 0 \\ &-(\beta_1^0\omega + \beta_2^0\Omega)\omega - (1 + lb^{-1}\cos\theta_0)K_0 = 0 \\ &rb^{-1}\omega\sin\theta_0 - (1 + lb^{-1}\cos\theta_0)\Omega = 0 \end{aligned} \tag{4.8}$$

A zero subscript denotes that the expression is calculated for the steady values (4.7) and $K_0 = (\partial\Theta/\partial q_3)_0$.

Assuming that $1 + lb^{-1}\cos\theta_0 \neq 0$ (for example, $l < b$), conditions (4.8) can be reduced to two independent relations

$$\Omega = \frac{r\sin\theta_0}{b + l\cos\theta_0}\omega, \quad \kappa\omega^2\sin^2\theta_0 = 0 \tag{4.9}$$

where

$$\kappa = m_T l_1 b + m_C l(b - d)$$

Relations (4.9) define one-parameter families of steady motions of the tricycle. These steady motions are the following:

(1) $$\sin\theta_0 = 0, \ \Omega = 0, \ \ \omega \neq 0 \ \text{ is an arbitrary quantity} \tag{4.10}$$

The steady motion is a rectilinear motion of the tricycle at an arbitrary speed. The direction of the motion is determined by the sign of ω. If $\theta_0 = 0$, then the third wheel is extended ahead with respect to the point A and, if $\theta = \pi$, the third wheel is extended to the rear with respect to the point A;

(2) $$\omega = \Omega = 0, \ \ \theta_0 \neq 0 \ \text{ is an arbitrary quantity} \tag{4.11}$$

The steady motion is a single parameter family of the equilibrium positions of the system;

(3) $$\begin{aligned} &\kappa = 0, \ \text{т.е.} \ m_T l_1 b + m_C l(b - d) = 0 \\ &\theta_0 \neq 0, \pi; \quad \omega \neq 0, \quad \Omega = \frac{r\sin\theta_0}{b + l\cos\theta_0}\omega \end{aligned} \tag{4.12}$$

In this case, when conditions (4.12), which relate the parameters of the system, are satisfied, such a steady motion of the tricycle occurs for which the plane of the third wheel is rotated through an angle θ_0 to the axis of symmetry of the trolley, the leading wheels rotate with an angular velocity $\omega/2$, the axis of symmetry of the trolley swivels with a velocity Ω and the projection of the centre of mass the trolley describes a circle in the horizontal plane.

We will now introduce deviation from the arbitrary steady motion

$$\dot{q}_1 = \omega + y_1, \quad \dot{q}_2 = \Omega + y_2, \quad q_3 = \theta_0 + z \tag{4.13}$$

The linearized equations of motion of system (4.6), (4.3) in the neighbourhood of the steady motion (4.7) have the form

$$A\dot{y} = P_1 y + V_1 z + F^{(1)}u^{(1)} + P_2^T F^{(2)} u^{(2)}, \quad \dot{z} = P_2 y + V_2 z \tag{4.14}$$

where

$$y = \begin{Vmatrix} y_1 \\ y_2 \end{Vmatrix}, \quad u^{(1)} = \begin{Vmatrix} u_1 \\ u_2 \end{Vmatrix}, \quad u^{(2)} = u_3$$

$$A = \begin{Vmatrix} a_{11}(\theta_0) & a_{12}(\theta_0) \\ a_{21}(\theta_0) & a_{22}(\theta_0) \end{Vmatrix}, \quad F^{(1)} = \begin{Vmatrix} 1 & 0 \\ 0 & b_{72} \end{Vmatrix}, \quad F^{(2)} = 1, \quad P_2 = \begin{Vmatrix} b_{31}(\theta_0) & b_{32}(\theta_0) \end{Vmatrix}$$

We will now consider the question of the stability of the rectilinear steady motion of the tricycle (4.10) when there are no controls.

In this case

$$P_1 = \begin{Vmatrix} 0 & 0 \\ 0 & -\omega\beta_2(\theta_0) \end{Vmatrix}, \quad V_1 = \begin{Vmatrix} 0 \\ \dfrac{lr^2\omega^2}{b^3}\varepsilon\gamma \end{Vmatrix}, \quad P_2 = \begin{Vmatrix} 0 & -\left(1 + \dfrac{l\varepsilon}{b}\right) \end{Vmatrix}$$

$$V_2 = \frac{r}{b}\omega\varepsilon \quad (\varepsilon = \cos\theta_0)$$

It is assumed here that the leading wheels are identical ($m_1 = m_2$, and then $J_4 = 0$, $S_2 = 0$)

The characteristic equation of system (4.14) has a single zero root, which corresponds to the existence of a one-dimensional manifold of steady motions. The remaining roots are found from the equation

$$\delta_0\lambda^2 + \delta_1\lambda + \delta_2 = 0 \tag{4.15}$$

According to the theorem on the stability of the steady motions of non-holonomic mechanical systems of the class being considered, which was proved in section 2, the conditions for the stability of the rectilinear motion of the tricycle have the form ($\delta_0 > 0$)

$$\delta_1 = \frac{\varepsilon\omega r}{b}\left[lr\left(\mu - \frac{\gamma}{b^2}\right) - a_{12}^1\right] > 0$$

$$\delta_2 = -\varepsilon\kappa > 0$$

Hence, in order for the rectilinear motion to be stable, the parameters of the system must satisfy the conditions

$$\varepsilon[m_T l_1 b + m_C l(b - d)] < 0$$

$$\omega\left\{m_T l_1(\varepsilon l_1 - b) + \varepsilon\left[(m_1 + m_2)a^2 + I_G + I_1 + I_2 + \frac{a^2}{r^2}(J_1 + J_2)\right] - \right.$$

$$\left. - m_C\left[bl + \frac{d^2 l}{b}(d + \varepsilon l) - 2dl\right] - \frac{l}{b}(I_3 + I_C)\right\} < 0 \tag{4.16}$$

The sign of δ_1 depends on the sign of ω, that is, the direction of the motion has a considerable effect on the stability of the given system, which is characteristic of non-holonomic mechanical systems [2, 3, 7, 12].

For a certain choice of the system parameters ($\delta_1 = 0$, $\delta_2 > 0$), the characteristic equation of system (4.15) will have a pair of imaginary roots, which indicates the possibility of the existence of periodic motions of the system when the values of the parameters are close to the above-mentioned values [13].

The conditions for the stability of the rectilinear motion, in the case of simpler model which does not take account of the inertia of the wheels, obtained earlier in [7] are a consequence of conditions (4.16).

We will now consider the question of the stability of the "rotational" motion (4.12) (case 3). For simplicity and clarity, we shall assume that $\theta_0 = \pi/2$.

The equations which have been linearized in the neighbourhood of this steady motion have the form of (4.14), where

$$P_1 = \omega\frac{lr}{b^2}\left\|\begin{matrix} -\frac{r^2}{b^2}\nu & \frac{r}{b}I_5 \\ \frac{r}{b}I_6 & -I_6 \end{matrix}\right\|, \quad P_2 = \left\|\begin{matrix} \frac{r}{b} & -1 \end{matrix}\right\|, \quad V_1 = \omega^2\frac{r^2l^2}{b^4}\left\|\begin{matrix} -\frac{r}{b}\nu \\ I_6 \end{matrix}\right\|,$$

$$V_2 = \omega\frac{lr}{b^2} \quad (I_6 = \nu + b^2\mu)$$

The characteristic equation also has a single zero root and the remaining roots are found from Eqs (4.15).

In this case $\delta_0 > 0$, $\delta_2 > 0$ and the stability condition $\delta_1 > 0$ takes the form

$$\omega\{a_{11}(\theta_0)I_6b^2 + a_{22}(\theta_0)[\nu r^2 - a_{11}(\theta_0)b^2]\} > 0 \tag{4.17}$$

As in the case of rectilinear motion, the conditions for the stability of the motion being considered in the case of the simpler model follow from the conditions (4.17) for the steady motions (4.12) [7].

We will now investigate the question of the possibility in principle, of stabilizing the steady motion of the tricycle, which corresponds to an analysis of the controllability of the system.

Generally speaking, the introduction of controls is possible for all the generalized coordinates $q_1, \dots, q_7$. In wheeled robots, control, as a rule, is achieved by introducing control moments with respect to the angular variables φ_1, φ_2, φ_3, θ, which characterize the rotations of the wheels (φ_1, φ_2, φ_3) and the strut (θ). These control moments correspond to the generalized control forces Q_j ($j = 1, 3, 6, 7$).

It is obvious that, when all the controls exist, the system is always controllable.

Control with respect to the coordinate ϑ is not considered ($Q_2 = 0$), since, for its introduction, an additional external moment of the forces has to be applied to the tricycle.

According to Theorem 2, the conditions of controllability have the form

$$\operatorname{rank} U = 3 \tag{4.18}$$

$$U = \left\|\begin{matrix} a_{11}\lambda_i - p_{11} & -p_{12} & -\nu_{11} & 1 & b_{31} & b_{61} & 0 \\ -p_{21} & a_{22}\lambda_i - p_{22} & -\nu_{12} & 0 & 0 & b_{62} & b_{72} \\ -b_{31} & -b_{32} & \lambda_i - \nu_{22} & 0 & b_{32} & 0 & 0 \end{matrix}\right\|, \quad i = 1, 2, 3$$

where $\lambda_1 = 0$ and λ_2 and λ_3 are found from Eq. (4.15).

The controllability of the system in the case of rectilinear motion for different methods of introducing the controls.

1. A control is introduced solely with respect to the coordinates φ_1 and φ_2 ($Q_1 \neq 0$, $Q_7 \neq 0$). It then follows from condition (4.18) that the system is controllable ($b_{32}b_{72} \neq 0$).
2. A control is introduced solely with respect to the coordinate φ_3. This control is formally equivalent to one of the versions of the introduction of a matched control with respect to the coordinates φ_1 and φ_2: $Q_{\varphi 1} = Q_{\varphi 2}$ ($Q_1 = 2Q_{\varphi 1}$, $Q_6 = 0$, $Q_7 = 0$). In these cases, the system is uncontrollable.

3. A control is introduced solely with respect to the coordinate θ ($Q_3 \neq 0$, $Q_1 = 0$, $Q_6 = 0$, $Q_7 = 0$). This control is formally equivalent to another version of the introduction of a matched control with respect to the coordinates φ_1 and φ_2: $Q_{\varphi 1} = -Q_{\varphi 2}$ ($Q_1 = 0$, $Q_3 = 0$, $Q_6 = 0$, $Q_7 = 2Q_{\varphi 1}$). In these cases, the system is also uncontrollable.

Controllability of the system in the case of "rotational" motion.
1. The system is controllable in this case when controls are introduced solely with respect to the coordinates φ_1 and φ_2 ($Q_1 \neq 0$, $Q_7 \neq 0$). If the control is matched: $Q_{\varphi 1} = Q_{\varphi 2}$ ($Q_1 = 2Q_{\varphi 1}$, $Q_6 = 0$, $Q_7 = 0$), the system is uncontrollable, as in the case of rectilinear motion.
2. If a control is introduced solely with respect to the coordinate φ_3 ($Q_1 = 0$, $Q_3 = 0$, $Q_6 \neq 0$, $Q_7 = 0$), the system is controllable.
3. If a control is introduced solely with respect to the coordinate θ, the system is controllable, unlike in the case of rectilinear motion.

The above analysis of the controllability enables one to construct algorithms for stabilizing the steady motions of the tricycle considered using the minimum number of controlling actions.

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